ON THE STABILIZATION OF UNSTABLE MOTIONS OF MECHANICAL SYSTEMS

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Consideration is given to the problem of control forces which stabilize an unstable motion of a holonomic mechanical system. Sufficient conditions for controllability and stabilization along one of the coordinates are derived. Conditions for observability of the system motion along one coordinate or one velocity are determined. The problem of optimum stabilization in the presence of incomplete feedback is considered.

1. Formulation of the problem. Let us consider a mechanical system, the states of which are described by the curvilinear coordinates $q_i(t)$ $(t \geqslant 0$ and $i = 1, \ldots, n)$. Let a control force u act on the system, where this force is related to the curvilinear coordinates q_i and velocities dq_i/dt by Equations

$$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = Q_i(t, q_1, \ldots, q_n, u) \qquad (i = 1, \ldots, n) \tag{1.1}$$

where T is the kinetic energy of the system, Q_i is the generalized force corresponding to the coordinate q_i .

Let there be given a motion $q_i = q_i^\circ(t)$ which results from (1.1) for $u \equiv 0$ and for certain initial conditions

$$q_i^\circ(0) = q_{i0} = \text{const}, \qquad (dq_i^\circ / dt)_{t=0} = q_{i0}^{(1)} = \text{const}$$

Let us assume that the motion $q_i = q_i^\circ(t)$ is unstable in the sense of Liapunov [1]. The problem is to determine the force u which stabilizes the motion $q_i^\circ(t)$. Let us construct the equations of disturbed motion [1] (p.21) at the vicinity of $q_i^\circ(t)$. Assuming that $s_i = q_i - q_i^\circ(t)$, then Equations (1.1) will be of the form

$$\frac{d}{dt}\frac{\partial T}{\partial s_i'} - \frac{\partial T}{\partial s_i} = S_i(t, s_1, \ldots, s_n, u) \quad (i = 1, \ldots, n) \tag{1.2}$$

which for $u \equiv 0$ possess the solution $s_i = 0$. We will consider two problems.

P r o b l e m 1.1 . Find the function

$$u = u\,(t,\ s_1,\ \ldots,\ s_n,\ s_1{}',\ \ldots,\ s_n{}') \tag{1.3}$$

such that the motion $s_i = 0$ be asymptotically stable in the sense of Liapunov on the strength of the equations of perturbed motion (1.2) and (1.3).

P r o b l e m 1.2 . Find the function u such that the motion $s_i(t) = 0$ be asymptotically stable on the strength of the equations of perturbed motion (1.2) and (1.3), and that at the same time the function

$$I = \int\limits_0^\infty G\,[t,\ s_1(t),\ \ldots,\ s_n(t),\ s_1{}'(t),\ \ldots,\ s_n{}'(t),\ u(t)]\,dt \tag{1.4}$$

be minimized along the motions $s_i(t)$ and $u(t)$ for the system (1.2) and (1.3).

Here G is a positive definite analytic function of s_i, $s_i{}'$, u for $t \geqslant 0$, and the following condition is satisfied

$$G\,(t,\ s_1,\ \ldots,\ s_n,\ s_1{}',\ \ldots,\ s_n{}',\ u) = \sum_{i,j=1}^{2n} d_{ij} z_i z_j + du^2 + \nu\,(t,\ z_1,\ \ldots,\ z_{2n},\ u)$$

$$(z_{2i-1} = s_i{}',\ z_{2i} = s_i)$$

Here the condition

$$|\,\nu\,(t,\ z_1,\ \ldots,\ z_{2n},\ u)\,| \leqslant \varepsilon\,(z_1{}^2 + \ldots + z_{2n}{}^2 + u^2)^{1+\alpha}$$

$$(\varepsilon > 0,\ \rho = (z_1{}^2 + \ldots + z_{2n}^2 + u^2)^{1/2} < \delta,\ \delta > 0,\ d > 0)$$

is fulfilled uniformly.

The quantity

$$\sum_{i,j=1}^{2n} d_{ij} z_i z_j + du^2$$

is a positive definite function.

2. The problem of stabilization. Let us assume that the linear approximation to the system is stationary and Equations (1.2) are of the form

$$\sum_{j=1}^{n} a_{ij} s_j'' = \sum_{j=1}^{n} b_{ij} s_j + b_j u + \gamma_j\,(t,\ s,\ s',\ u) \quad (i = 1,\ \ldots,\ n)\ \left(S = \sum_{i,j=1}^{n} a_{ij} s_i s_j\right) \tag{2.1}$$

where a_{ij}, b_{ij}, b_i are constants, and S is a positive definite form, $b_{ij} = b_{ji}$. It is assumed that condition

$$\gamma_i\,(t,\ s_1,\ \ldots,\ s_n,\ s_1{}',\ \ldots,\ s_n{}',\ u)\,| \leqslant \varepsilon\rho^2 \qquad (\varepsilon > 0,\ \rho < \delta,\ \delta > 0,\ i = 1, \ldots, n) \tag{2.2}$$

is fulfilled uniformly.

Without loss of generality, we may assume that $b_1 \neq 0$, $b_i = 0$ $(i = 2,\ldots,n)$. If $b_i = 0$ and $b_k \neq 0$ for $i \neq k$ then it will be said that the system is subject to a force along the kth coordinate.

The linear approximation for Equation (1.2) will be of the form

$$\sum_{i=1}^{n} a_{1i} s_i'' = \sum_{i=1}^{n} b_{1i} s_i + u, \qquad \sum_{i=1}^{n} a_{ji} s_i'' = \sum_{i=1}^{n} b_{ji} s_i \qquad (j = 2, \ldots, n) \quad (2.3)$$

With the aid of a nonsingular linear transformation [2] (p.97)

$$s_i = \beta_{i1} y_1 + \ldots + \beta_{in} y_n$$

the system will be reduced to normal coordinates

$$y_i'' = \lambda_i y_i + \alpha_{1i} u \qquad (i = 1, \ldots, n) \quad (2.4)$$

Here y_i are the normal coordinates and the real numbers λ_i are the roots of Equation

$$|a_{ij}\lambda - b_{ij}| = 0 \quad (2.5)$$

The numbers α_{ki} satisfy Equations

$$\sum_{k=1}^{n} (\alpha_{kj}\lambda_j - b_{kj})\alpha_{ki} = 0 \qquad (i,j = 1,\ldots,n) \quad (2.6)$$

The system (2.4) is replaced by the system

$$x_{2i-1}' = \lambda_i x_{2i} + \alpha_{1i} u, \qquad x_{2i}' = x_{2i-1} \quad (2.7)$$
$$(x_{2i-1} = y_i', \ x_{2i} = y_i; \ i = 1, \ldots, n)$$

Let us formulate the conditions for solvability of problem 1.1 . A sufficient condition for solvability of problem 1.1 is the following [3 and 4]. The system of vectors

$$A, BA, \ldots, B^{2n-1}A \quad (2.8)$$

must be linearly independent, where

$$A = \left\| \begin{matrix} \alpha_{11} \\ 0 \\ \ldots \\ \alpha_{1n} \\ 0 \end{matrix} \right\|, \qquad B = \left\| \begin{matrix} 0 & \lambda_1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & \lambda_n \\ 0 & 0 & \ldots & 1 & 0 \end{matrix} \right\| \quad (2.9)$$

It follows from the requirement of linear independence of vectors (2.8) that

$$\Delta = \begin{vmatrix} \alpha_{11} & 0 & \ldots & \alpha_{11}\lambda_1^{n-1} & 0 \\ 0 & \alpha_{11} & \ldots & 0 & \alpha_{11}\lambda_1^{n-1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \alpha_{1n} & 0 & \ldots & \alpha_{1n}\lambda_n^{n-1} & 0 \\ 1 & \alpha_{1n} & \ldots & 0 & \alpha_{1n}\lambda_1^{n-1} \end{vmatrix} \neq 0 \quad (2.10)$$

We have

$$\Delta = \alpha_{11}^2 \ldots \alpha_{1n}^2 (\lambda_2 - \lambda_1)^2 \ldots (\lambda_n - \lambda_1)^2 \ldots (\lambda_n - \lambda_{n-1})^2$$

Consequently, condition (2.10) can be expressed as

$$1. \quad \lambda_i \neq \lambda_j, \qquad 2. \quad \alpha_{1i} \neq 0 \qquad (i, j = 1, \ldots, n;\ i \neq j) \tag{2.11}$$

Conditions (2.11) are the conditions for controllability [5] of the linear system (2.7), i.e. when (2.11) is fulfilled for any $T > 0$ and any initial point x^0, there exists [5 and 6] a control $u(t)$ $(0 \leqslant t \leqslant T)$, which translates the system (2.7) from the point $x = x^0$ to the point $x = 0$ in time T. Furthermore, with conditions (2.11), we can indicate such a neighborhood of the point $x = 0$ where there also is a control $u(t)$ for the nonlinear system (1.2) for each point x^0 from this neighborhood, which will transfer the system (1.2) into the state $x = 0$ for finite time T. According to [7], conditions (2.11) allow the system (2.7) to be transferred from any point x^0 into a point $x = 0$ in time T also by the impulse control

$$u = \eta_1 \delta (t - t_1) + \ldots + \eta_k \delta (t - t_k)$$

Here t_j are instants when the function

$$\varphi (t) = | lF^{-1} (t) \alpha |, \qquad \alpha = \{\alpha_{11}, 0, \ldots, \alpha_{1n}, 0\}$$

has a strict maximum, $F(t)$ is the fundamental matrix of the system (2.7), and $l = \{l_1, \ldots, l_{2n}\}$ is a solution of the problem $\min_l \max_\tau |lF^{-1}(t)\alpha|$ for $(x_0 l) = -1$. It can be verified that under the conditions (2.11) and $\lambda_j \neq 0$, the function $\varphi(t)$ for any choice of l can have only isolated maximums. Thus, under conditions (2.11) and $\lambda_j \neq 0$ we can construct a sequence of force impulses directed along the first coordinate such that the system (2.7) will be transferred by these impulses from point x^0 into the point $x = 0$.

Let conditions (2.11) be fulfilled, then we can find the function

$$u = p_1 x_1 + \ldots + p_{2n} x_{2n} \tag{2.12}$$

such that the system (2.7), (2.12) will be asymptotically stable. Consequently, according to a Liapunov theorem [1] (p.127), the system (1.2), (1.3) will also be asymptotically stable.

Let conditions (2.11) not be fulfilled. We will consider two cases.

Case 1. Let

$$\lambda_i \neq \lambda_j, \quad \alpha_{1i_k} = 0 \quad (k = 1, \ldots, p), \qquad p < n, \quad \alpha_{1j} \neq 0, \quad j \neq i_k$$

Then, if among the numbers λ_{i_k} there is at least one positive number, the system (2.7) will have positive numbers among the roots of its characteristic equation for any choice of u (2.12). Consequently, according to a theorem by Liapunov, the system (2.7) is unstable for any choice of u (2.12).

If, on the other hand, all numbers λ_{i_k} are negative, then (2.7) can be considered as two independent systems of the type

$$x'_{2i_k-1} = \lambda_{i_k} x_{2i_k}, \qquad x'_{2i_k} = x_{2i_k-1} \qquad (k = 1, \ldots, p) \tag{2.13}$$

$$x'_{2i_k-1} = \lambda_i x_{2j} + \alpha_{1j} u, \qquad x'_{2j} = x_{2j-1} \qquad (j \neq i_k) \tag{2.14}$$

For the system (2.14) conditions (2.11) are fulfilled, and therefore, we can choose

$$u = p_1 x_1 + \ldots + p_{2n} x_{2n} \qquad (j \neq i_k) \tag{2.15}$$

such that the system (2.14), (2.15) be asymptotically stable. For such a choice of u the system of first approximation (2.7), (2.12) is stable and there are imaginary values among its characteristic numbers. The stability of the complete system is then determined by the terms of higher order of smallness [1] (p.137) [8].

If all $\lambda_{i_k} \leqslant 0$ and if at least one $\lambda_{i_k} = 0$, then again a critical case arises; the stability of the complete system is again determined by the same terms.

Case 2. Let

$$\lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+p} = \lambda, \qquad \alpha_{1j} \neq 0 \tag{2.16}$$

for at least one $j = k, \ldots, k + p$. Without loss of generality we will assume that

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{p+1} = \lambda, \qquad \alpha_{1p+1} \neq 0 \tag{2.17}$$

Let us transform the coordinates

$$z_j = \sum_{k=1}^{p+1} c_{jk} y_k, \qquad z_i = y_i \qquad (j = 1, \ldots, p+1;\ i = p+2, \ldots, n) \tag{2.18}$$

and require that

$$\sum_{k=1}^{p+1} c_{ik} \alpha_{1k} = 0 \qquad (i = 1, \ldots, p), \qquad \sum_{k=1}^{p+1} c_{p+1k} \alpha_{1k} \neq 0 \tag{2.19}$$

The system (2.4) is reduced to the form

$$z_i'' = \lambda z_i \qquad (i = 1, \ldots, p), \qquad z''_{p+1} = \lambda z_{p+1} + \sum_{k=1}^{p+1} c_{p+1k} \alpha_{1k} u \tag{2.20}$$

$$z_i'' = \lambda_i z_i + \alpha_{1i} u \qquad (i = p + 2, \ldots, n)$$

It follows from (2.20) that if $\lambda > 0$, the linear approximation (2.7), (2.12) is unstable, consequently [1] (p.128), the complete system is also unstable.

If $\lambda \leqslant 0$, then the stability of the system is determined by the terms of higher order of smallness.

It is known that problem 1.2 is solvable if problem 1.1 is solvable in the linear approximation [9].

Thus, the following assertion is valid.

T h e o r e m 2.1 . If conditions (2.11) are fulfilled, then problems 1.1 and 1.2 are solvable. If conditions (2.11) are not fulfilled and

1) $\alpha_{1i_k} = 0$ $(k = 1, \ldots, p)$ and at least one of the numbers $\lambda_{i_k} > 0$, then problems 1.1 and 1.2 are not solvable for $\lambda_{i_k} \leqslant 0$, but $\lambda_i \neq \lambda_j$, $i \neq i_k$, $j = i_k$ yields a critical case, i.e. the possibility of solvability of problem 1.1 depends on the terms of higher order of smallness.

2) if $\lambda_1 = \lambda_2 = \ldots = \lambda_{p+1} = \lambda$, but $\alpha_{1p+1} \neq 0$, then problems 1.1 and 1.2 do not possess a solution for $\lambda > 0$ and are reduced to critical cases if $\lambda \leqslant 0$, $\lambda_i \neq \lambda_j$, $\alpha_{1i} \neq 0$ for $i > p + 1$.

Let us consider now the linear approximation to problem 1.2 . If it is assumed [10 and 11] that the functional (1.4) in the first approximation becomes

$$I_2 = \int_0^\infty \Big[\sum_{i,k=1}^{2n} d_{ik} x_i x_k + du^2 \Big] dt \tag{2.21}$$

then by minimizing it we obtain [12] the equations for u (2.12) and the Liapunov function V which ensures the asymptotic stability of the system (2.7), (2.12) in the form

$$\sum_{k=1}^{n} \Big[\frac{\partial V(x)}{\partial x_{2k-1}} (\lambda_k x_{2k} + \alpha_{1k} u) + \frac{\partial V(x)}{\partial x_{2k}} x_{2k-1} \Big] + \sum_{i,k=1}^{2n} d_{ik} x_i x_k + du^2 = 0 \tag{2.22}$$

$$u = -\frac{1}{2d} \sum_{k=1}^{n} \frac{\partial V}{\partial x_{2k-1}} \alpha_{1k}$$

The V function can be sought as a quadratic form, and the coefficients determined by equating to zero the coefficients of terms in (2.22).

The obtained algebraic equations have a solution then and only then when there exists a control $u = p_1 x_1 + \ldots + p_{2n} x_{2n}$, satisfying the conditions of the problem 1.1 in the linear approximation. This indicates a way for computing the control.

3. The problem of observation in the linear approximation.

P r o b l e m 3.1 . Find a $2n \times n$ matrix $V(\vartheta)$ such that

$$\int_{t-\tau}^{t} V(\vartheta) \begin{Vmatrix} \xi(\vartheta) \\ u(\vartheta) \end{Vmatrix} d\vartheta = \begin{Vmatrix} x_1(t) \\ \cdots \\ x_{2n}(t) \end{Vmatrix}, \qquad \xi(\vartheta) = \sum_{i=1}^{2n} c_i x_i(\vartheta) \qquad (-\tau \leqslant \vartheta \leqslant 0) \tag{3.1}$$

where $x_i(\vartheta)$ will be solutions of the system (2.7) and $u(\vartheta)$ is determined by (2.12).

It is known that the solution of problem 3.1 is given by Lemma 4.2 [9] under the conditions which in the stationary case assume the following form [5, 9 and 13]. The system (2.7) is observable then and only then when the vector system

$$C,\ B^*C,\ \ldots,\ B^{*2n-1}C \tag{3.2}$$

is linearly independent. Here

$$C = (c_1, \ldots, c_{2n}), \qquad B^* = \begin{Vmatrix} 0 & 1 & \ldots & 0 & 0 \\ \lambda_1 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & \lambda_n & 0 \end{Vmatrix}$$

Let us consider the case $C=(c_1, 0, c_3, 0, \ldots, c_{2n-1}, 0)$, which corresponds to the observation of the system along a certain velocity, and the case when $C = (0, c_2, \ldots, 0, c_{2n})$ which corresponds to observation along a certain curvilinear coordinate.

The conditions of observability in the first case will be

$$c_{2i-1} \neq 0, \quad \lambda_i \neq \lambda_j \quad (i, j = 1, \ldots, n), \quad \lambda_i \neq 0 \tag{3.3}$$

and in the second

$$c_{2i} \neq 0, \qquad \lambda_i \neq \lambda_j \qquad (i, j = 1, \ldots, n;\ i \neq j) \tag{3.4}$$

When conditions (3.3) or (3.4) are fulfilled, the matrix $V(\vartheta)$ is determined from Formula (4.20) of [9] or in other possible forms indicated in the paper cited.

Note, in particular, that under conditions (3.3) and (3.4) the first column of the $V(\vartheta)$ matrix can be chosen ([9], Equation (4.30)) in the form of a linear combination of δ-functions

$$V_{i1}(\vartheta) = \sum_{k=1}^{m} \alpha_k{}^i\ \delta(\vartheta - \tau_k{}^i)$$

for a finite number of instants $\tau_k{}^i$.

This means that at a given time the state $x_1(t), \ldots, x_{2n}(t)$ of the system $dx_{2i-1}/dt = \lambda_i x_{2i}$, $dx_{2i}/dt = x_{2i-1}$ under the conditions (3.3) or (3.4) can be restored by measurements of the quantity $\xi(t+\vartheta)$ at discrete instants of time $t_k{}^i = t - \tau_k{}^i$. The reasoning of Sections 2 and 3 justifies the following assertion.

The mechanical system (2.3) is observable in the quantity

$$\xi = \sum_{i=1}^{n} c_{2i} x_{2i}$$

then and only then when it is controlled by a force directed in the space $\{x_i\}$ $(i = 1, \ldots, n)$ along the vector C. The system is observable in velocity

$$\xi' = \sum_{i=1}^{n} c_{2i-1} x_{2i-1}$$

then and only then when it is controlled by a force directed in the space $\{x_i\}$ along the vector C, and when all $\lambda_j \neq 0$.

The letters $\zeta_i (i = 1, \ldots, n)$ will be used for notation.

Let $\zeta_i (i = 1, \ldots, n)$ be curvilinear coordinates in which the kinetic energy is expressed as a sum of squares. Then, the above assertion is formulated as follows.

The mechanical system is observable on a coordinate $\xi = \zeta_i$ then and only then when it is controllable by a force along this coordinate. The system is observable along the velocity $\xi' = \zeta_i'$ then and only then when it is controlled by a force along the coordinate ζ_i and all $\lambda_j \neq 0$. This represents the concrete expression of the dualty principle between control and observation [6 and 13] for the considered mechanical systems.

4. Solution of problems 1.1 and 1.2 with incomplete information. Let us suppose that it is impossible to measure $x_{si} (i = 1, \ldots, 2n)$ at each instant of time but that it is possible to measure only certain functions of them $w_i = \varphi_i (x_1, \ldots, x_{2n})$ which are not solvable uniquely with respect to x_i and which satisfy the condition $\varphi_i (0, \ldots, 0) = 0$. It is required to find a control satisfying the conditions of problems 1.1 and 1.2.

Following [9], we seek the control of the form

$$\frac{du}{dt} = U\, [w_1 (t + \vartheta), \ldots, w_l (t + \vartheta), u (t + \vartheta)] \qquad (4.1)$$

where U is the functional defined on the continuous functions $w_i (\vartheta)$, $u (\vartheta)$ $(-\tau \leqslant \vartheta \leqslant 0,\ \tau = \text{const} > 0,\ i = 1, \ldots, l)$. The solution of the linear problem, corresponding to problem 1.1, exists under the conditions indicated in [9] and be determined by the equality (4.1) of [9]. These conditions coincide with conditions (2.11) in the present case if the observation is carried out along the coordinate, or with conditions (2.11) and (3.3), if the velocity is observed. The solvability of problem 1.2 follows from the solvability of problem 1.1 in the linear approximation. Also, the solvability of problems 1.1 and 1.2 in the linear approximation indicates the solvability of the corresponding nonlinear problems [9].

T h e o r e m 4.1. Let the system (1.2) be observed along the coordinate

$$w_i = c_{i1} x_1 + \ldots + c_{in} x_n + \mu_i (x, x')$$

where μ_i and μ_i^* are terms of higher order of smallness. If condition (2.11) is fulfilled, then the motion $s_1 = \ldots = s_n = 0$ can be stabilized by the control

$$\frac{du}{dt} = U\,[w\,(t + \vartheta),\, u\,(t + \vartheta)] \tag{4.2}$$

Let the system (1.2) be observed in velocity

$$w_i = c_{i1}x_1' + \ldots + c_{in}x_n' + \mu_i^*\,(x,\, x')$$

If the conditions (2.11) and (3.3) are fulfilled, then the motion $s_1' = \ldots s_n' = 0$ can be stabilized by the control (4.2).

5. Example. Let us suppose that there are n rods of lengths $l_1, \ldots, l_n$ connected by hinges (see Fig.1). At the rod attachment points and at the free end, there are point masses $m_1, \ldots, m_n$. The rod masses are neglected. We will assume that the system is in the vertical plane. The initial deviations from the vertical and the velocities of the points of the system are regarded as small quantities.

Let the force be applied to the kth point having a horizontal direction and lying in the given vertical plane. Let us determine the possibility of stabilization in the sense of problem 1.1 and observation in the sense of problem 3.1.

Let us choose as independent coordinates the deviations x_i of the points m_i from the vertical (Fig.1). In the first approximation we have

$$2T = \sum_{i=i}^{n} m_i x_i'^2, \quad 2V = -g \sum_{i=1}^{n} m_i \sum_{k=1}^{i} \frac{1}{l_k}(x_k - x_{k-1})^2 \tag{5.1}$$

Here T and V are the kinetic and the potential energies. The equations of motion are of the form

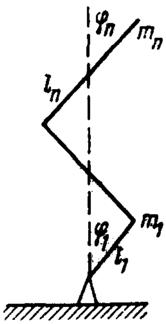

Fig.1

$$\begin{aligned} x''_1 &= \alpha_1 x_1 - \beta_1 x_2 \\ x_2'' &= -\gamma_1 x_1 + \alpha_2 x_2 - \beta_2 x_3 \\ &\ldots\ldots\ldots\ldots \\ x_n'' &= -\gamma_{n-1} x_{n-1} + \alpha_n x_n \end{aligned} \tag{5.2}$$

where

$$\alpha_i = \frac{g}{m_i l_i} \sum_{k=i}^{n} m_k + \frac{g}{m_i l_{i+1}} \sum_{k=i+1}^{n} m_k$$

$$\beta_i = \frac{g}{m_i l_{i+1}} \sum_{k=i+1}^{n} m_k, \qquad \gamma_{i-1} = \frac{g}{m_i l_i} \sum_{k=i}^{n} m_k \tag{5.3}$$

Equations (2.5) are, in this case, of the form

$$\begin{vmatrix} \alpha_1 - \lambda & -\beta_1 & 0 & \ldots & 0 & 0 \\ -\gamma_1 & \alpha_2 - \lambda & -\beta_2 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -\gamma_2 & \alpha_3 - \lambda & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \alpha_{n-1} - \lambda & -\beta_{n-1} \\ 0 & 0 & 0 & \ldots & -\gamma_{n-1} & \alpha_n - \lambda \end{vmatrix} = 0 \tag{5.4}$$

It follows from (5.3) that $\beta_i \gamma_i > 0$. Consequently, [14] (p.82) the roots of Equation (5.4) are different and no coordinate of any eigenvector for the matrix considered can be zero; therefore, $\lambda_i \neq \lambda_j$, $\alpha_{kj} \neq 0$, $\lambda_i \neq 0$ $(k,\, i,\, j = 1, \ldots, n;\; i \neq j)$ (V is negative definite).

This means that the system considered is controllable along any coordinate x_i and is observable along any coordinate x_i and the velocity x_i'.

Consequently, the following conclusions are valid.

1. The system (Fig.1) can be stabilized by the force

$$u\,(x_1, \ldots, x_n, x_1', \ldots, x_u')$$

2. The system (Fig.1) can be observed along the coordinate

$$w = x_i + \mu_i\,(x_1, \ldots, x_n)$$

or this system can be observed along velocity

$$w = x_i' + \mu_i^*\,(x_1', \ldots, x_n')$$

and stabilized by the control (4.2)

3. The system (Fig.1) in the linear approximation can be reduced to the state $x_1 = 0$ in the finite time T by application of a sequence of impulses of the force u.

N o t e . The considered rod system is a Sturm system [14]. The above derived conclusions are applicable to Sturm systems in general.

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